This document contains solutions to selected problems in Steven Weinberg's "The Quantum Theory of Fields: Volume I". The solutions are original work from Hong-Yi Zhang, Siyang Ling, Jiazhao Lin, and Ray Hagimoto. We decided to study Weinberg's QFT Vol. I as a group and the following solutions are the result of this effort.

In the following equations from Weinberg are written with periods delimiting the chapter, subsection, and equation number e.g. (5.2.21). Equations delimited by hyphens (-) refer to equations in this PDF, e.g. (2-1-2).

Contents

Chapter 2

Problem 2

Suppose that observer $\mathcal{O}$ sees a photon with momentum $\boldsymbol{p}$ in the y-direction and polarisation vector in the z-direction. A second observer $\mathcal{O}'$ moves relative to the first with velocity $\boldsymbol{v}$ in the z-direction. How does $\mathcal{O}'$ describe the same photon?

Solution

The reference frames of $\mathcal{O}$ and $\mathcal{O}'$ are related by a Lorentz transformation Λ. This means that for any state ray $\mathscr{R}$ seen in the frame of $\mathcal{O}$, $\mathcal{O}'$ sees the same particle in the equivalent state $\mathscr{R}' = T(\Lambda))\mathscr{R}$.

As seen by $\mathcal{O}$, the photon has four-momentum $P = (0, p, 0, p)$ (where we follow Weinberg's convention of using the ordering (P^1, P^2, P^3, P^0)) and helicity $\sigma = +1$. The physical state is the ray $\mathscr{R} \equiv \mathscr{R}_{P,\sigma}$. Any vector $\Psi = \Psi_{P,\sigma} \in \mathscr{R}_{P,\sigma}$ sufficiently describes this state. Since $\mathcal{O}'$ is moving relative to $\mathcal{O}$ with velocity $\boldsymbol{v} \equiv (0, 0, v)$, their reference frames are related by a Lorentz boost in the z-direction, $\Lambda = B(v)$. We therefore compute $U(\Lambda)\Psi_{P,\sigma}$. The appropriate relation is given in eqn. (2.5.42) as

$$U(\Lambda)\Psi_{P,\sigma} = \sqrt{\frac{(\Lambda P)^0}{P^0}} e^{i\sigma\theta(\Lambda,P)} \Psi_{\Lambda P,\sigma}.$$

where

$$\Lambda^\mu{}_\nu P^\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma(v) & -\gamma(v)\,v \\ 0 & 0 & -\gamma(v)\,v & \gamma(v) \end{pmatrix} \begin{pmatrix} 0 \\ p \\ 0 \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ p \\ -\gamma(v)\,v\,p \\ \gamma(v)\,p \end{pmatrix} \tag{2-2-1}$$

If Ψ had been a superposition of helicity eigenstates the phase factor $e^{i\sigma\theta(\Lambda,P)}$ would be important, but because physical states are represented by rays, we are perfectly fine saying that $\mathcal{O}'$ sees the particle in the state $\sqrt{\frac{(\Lambda P)^0}{P^0}}\Psi_{\Lambda P,\sigma}$. Qualitatively, this just means that $\mathcal{O}'$ would describe the photon as having four-momentum $P' = \Lambda P = (0, p, -\gamma(v)\,v\,p, \gamma(v)\,p)$ and helicity $\sigma' = \sigma = +1$.

Since superposition states are quite ubiquitous (e.g. circularly polarised photons) it is worth showing how one would compute $\theta(\Lambda, P)$ even though it is not of critical importance in this problem. Recall that $\theta(\Lambda, P)$ is defined by writing the little group element $W(\Lambda, P)$ in the form

$$W(\Lambda, P) = S(\alpha, \beta)R(\theta)$$

where R is a rotation about the 3-axis, and $S(\alpha, \beta)$ is a Lorentz transformation whose matrix elements in a basis of four-vectors is given by (2.5.26). On the other hand, $W(\Lambda, P)$ may also be written in terms of the Wigner rotation

$$W(\Lambda, P) = L^{-1}(\Lambda P)\,\Lambda\,L(P).$$

where by convention we choose the standard Lorentz transformations L to be given by eqn. (2.5.44). Critically, we have $S^{-1}(\alpha,\beta)W(\Lambda,P) = R(\theta)$. The information we can easily extract from the question prompt is that $\Lambda^\mu{}_\nu$ is given by the 4×4 matrix appearing in eqn. (2-2-1). For $L(P)^\mu{}_\nu$ we have*

$$\begin{aligned} L(P)^\mu{}_\nu &= R(\hat{\boldsymbol{p}})B(p/\kappa) = R_3(\pi/2)R_2(\pi/2)B(p/\kappa) \\ &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{(p/\kappa)^2+1}{2(p/\kappa)} & \frac{(p/\kappa)^2-1}{2(p/\kappa)} \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \frac{(p/\kappa)^2-1}{2(p/\kappa)} & \frac{(p/\kappa)^2+1}{2(p/\kappa)} \end{pmatrix} \end{aligned}$$

and for $L(\Lambda P)^\mu{}_\nu$ we have

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -v & 0 & \frac{1}{\gamma}\frac{(\gamma p/\kappa)^2+1}{2(\gamma p/\kappa)} & \frac{1}{2\gamma}\left(\frac{\gamma p}{\kappa} - \left(\frac{\gamma p}{\kappa}\right)^{-1}\right) \\ -1/\gamma & 0 & -v\frac{(\gamma p/\kappa)^2+1}{2(\gamma p/\kappa)} & -v\frac{(\gamma p/\kappa)^2-1}{2(\gamma p/\kappa)} \\ 0 & 0 & \frac{(\gamma p/\kappa)^2-1}{2(\gamma p/\kappa)} & \frac{(\gamma p/\kappa)^2+1}{2(\gamma p/\kappa)} \end{pmatrix}.$$

Thus $W^\mu{}_\nu = \left(L^{-1}(\Lambda P)\Lambda L(P)\right)^\mu{}_\nu$ is

$$\begin{pmatrix} 1 & 0 & -v\frac{\kappa}{p} & v\frac{\kappa}{p} \\ 0 & 1 & 0 & 0 \\ v\frac{\kappa}{p} & 0 & 1-\frac{1}{2}\left(v\frac{\kappa}{p}\right)^2 & \frac{1}{2}\left(v\frac{\kappa}{p}\right)^2 \\ v\frac{\kappa}{p} & 0 & -\frac{1}{2}\left(v\frac{\kappa}{p}\right)^2 & 1+\frac{1}{2}\left(v\frac{\kappa}{p}\right)^2 \end{pmatrix}.$$

If we use the fact that $W = S(\alpha,\beta)R(\theta)$ then, using the fact that the last two columns of $S(\alpha,\beta)^\mu{}_\nu$ are unchanged upon multiplication with $R(\theta)^\mu{}_\nu$ then we can simply read off the values $\alpha = v\kappa/p$ and $\beta = 0$ from $W^\mu{}_\nu$†. Then computing $\left(S(\alpha,\beta)^{-1}W\right)^\mu{}_\nu$ gives

$$\left(S(\alpha,\beta)^{-1}W\right)^\mu{}_\nu = R(\theta)^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

from which we deduce that $\theta(\Lambda, P) = 0$.

*here P is the 4-momentum and $\hat{\boldsymbol{p}} = \boldsymbol{p}/p$ where $p = |\boldsymbol{p}|$ and $\boldsymbol{p}$ is the 3-vector part of P.

†Mathematica notebook showing how the calculation was performed available at http://rayhagimoto.xyz/notes/QFT/weinberg-solutions-2-2.nb

Problem 5

> Consider physics in two space and one time dimensions, assuming invariance under a 'Lorentz' group $SO(2,1)$. How would you describe the spin states of a single *massive* particle? How do they behave under Lorentz transformations? What about the inversions **P** and **T**?

Solution

A massive particle in $1+2$ spacetime has standard momentum:

$$k^\mu = (0, 0, M) \tag{2-5-1}$$

The little group is $SO(2)$. Under Lorentz transform Λ, an arbitrary state $\Psi_{p,\sigma}$ (could be a multi-particle state) transforms as:

$$U(\Lambda)\Psi_{p,\sigma} = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, p))\Psi_{\Lambda p,\sigma'} \tag{2-5-2}$$

The matrices $D_{\sigma'\sigma}(W)$ are given by a unitary representation of $SO(2)$:

$$D : SO(2) \to SU(N) \tag{2-5-3}$$

We shall parametrize $SO(2)$ by the rotation angle θ. To find the spin states of a *single* particle, we should find the irreducible representations of $SO(2)$. Since $SO(2)$ is 1-dimensional, we can write D for the infinitesimal rotation $R(\theta)$ as:

$$D = 1 + i\theta t + \mathcal{O}\big(\theta^2\big) \tag{2-5-4}$$

Here, t is a Hermitian matrix, so we can use spectral decomposition to find an orthonormal basis in which it is diagonal. We can thus assume WLOG that t is diagonal: (by redefining $\Psi_{p,\sigma}$ using this orthonormal basis)

$$t = \mathrm{diag}(n_1, n_2, \ldots, n_N) \tag{2-5-5}$$

So:

$$D(R(\theta)) = \exp(i\theta \mathrm{diag}(n_1, n_2, \ldots, n_N)) = \mathrm{diag}(e^{in_1\theta}, e^{in_2\theta}, \ldots, e^{in_N\theta}) \tag{2-5-6}$$

The representation is decomposed into a direct sum of N 1-dimensional representations. An (1-dimensional) irreducible representation has the form:

$$D(R(\theta)) = e^{in\theta}, \tag{2-5-7}$$

where n is an integer. Under Lorentz transform, a single particle transforms as:

$$U(\Lambda)\Psi_{p,n} = \sqrt{\frac{(\Lambda p)^0}{p^0}} e^{in\theta(W(\Lambda,p))}\Psi_{\Lambda p,n} \tag{2-5-8}$$

Note that σ is replaced with n.

Now we consider the transformation of $\Psi_{k,n}$ under parity P and time-reversal T. We know:

$$\begin{aligned} HP\Psi_{k,n} &= PHP^{-1}P\Psi_{k,n} = PH\Psi_{k,n} = MP\Psi_{k,n} \\ \mathbf{P}P\Psi_{k,n} &= -P\mathbf{P}P^{-1}P\Psi_{k,n} = P\mathbf{P}\Psi_{k,n} = 0 \\ tP\Psi_{k,n} &= P(P^{-1}J^{12}P)\Psi_{k,n} = Pt\Psi_{k,n} = nP\Psi_{k,n} \end{aligned} \tag{2-5-9}$$

If the particle with $H = M$, $\mathbf{P} = 0$ and $t = n$ has no degeneracy, then $\Psi_{k,n}$ and $P\Psi_{k,n}$ are the same state, and:

$$\begin{aligned} &P\Psi_{k,n} = \eta\Psi_{k,n} \\ &P\Psi_{p,n} = \sqrt{M/p^0}(PU(L(p))P^{-1})P\Psi_{k,n} = \sqrt{M/p^0}U(L(\mathcal{P}p))P\Psi_{k,n} = \eta\Psi_{\mathcal{P}p,n} \end{aligned} \tag{2-5-10}$$

where $|\eta| = 1$ is some phase. Similarly:

$$\begin{aligned} &HT\Psi_{k,n} = MT\Psi_{k,n} \\ &\mathbf{P}T\Psi_{k,n} = 0 \\ &tT\Psi k,n = -nT\Psi_{k,n} \end{aligned} \tag{2-5-11}$$

So:

$$\begin{aligned} &T\Psi_{k,n} = \zeta\Psi_{k,-n} \\ &T\Psi_{p,n} = \sqrt{M/p^0}(TU(L(p))T^{-1})T\Psi_{k,n} = \sqrt{M/p^0}U(L(\mathcal{P}p))T\Psi_{k,n} = \zeta\Psi_{\mathcal{P}p,-n} \end{aligned} \tag{2-5-12}$$

where $|\zeta| = 1$ is some phase. However, this phase has no physical significance since it can be eliminated by redefining $\Psi_{k,n}$:

$$T\zeta^{1/2}\Psi_{k,n} = \zeta^{*1/2}T\Psi_{k,n} = \zeta^{1/2}\Psi_{k,n} \tag{2-5-13}$$

Chapter 3

Problem 3

Express the differential cross-section for two-body scattering in the rest *laboratory* frame, in which one of the two particles is initially at rest, in terms of kinematic variables and the matrix element $M_{\beta\alpha}$. (Derive the result directly, without using the results derived in this chapter for the differential cross-section in the center-of-mass frame.)

Solution

In two-body scattering we have a general two-body 'in' state $\mathbf{\Psi}_\alpha^+$ and a general two-body 'out' state $\mathbf{\Psi}_\beta^-$.

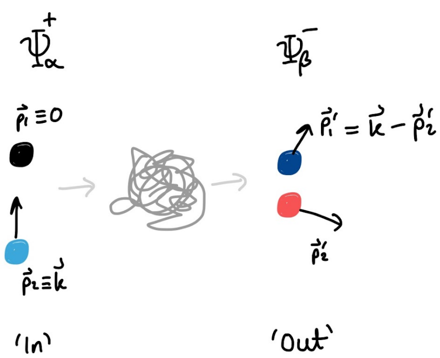

Figure 1: Two-body interaction ($N_\alpha = N_\beta = 2$) as viewed in the 'laboratory' frame. Two particles scatter, undergoing some complicated interaction to produce a general 'out' state of two particles. Colours represent the fact that the particles may, in general, be different species.

From (3.4.15) we have

$$\mathrm{d}\sigma(\alpha \to \beta) = (2\pi)^4 u_\alpha^{-1} |M_{\beta\alpha}|^2 \delta^4(p_\beta - p_\alpha)\, \mathrm{d}\beta$$

where $\mathrm{d}\beta \equiv \mathrm{d}^3\boldsymbol{p}_1'\, \mathrm{d}^3\boldsymbol{p}_2'$. In the *laboratory frame* one of the particles is at rest. We write this as

$$\boldsymbol{p}_1 \equiv 0, \qquad \boldsymbol{p}_2 \equiv \boldsymbol{k}.$$

Hence,

$$\begin{aligned}\delta^4(p_\beta - p_\alpha)\, \mathrm{d}\beta &= \delta^3(\boldsymbol{p}_1' + \boldsymbol{p}_2' - \boldsymbol{p}_1 - \boldsymbol{p}_2)\, \delta(E_1' + E_2' - E)\, \mathrm{d}^3\boldsymbol{p}_1'\, \mathrm{d}^3\boldsymbol{p}_2' \\ &= \delta^3(\boldsymbol{p}_1' + \boldsymbol{p}_2' - \boldsymbol{k})\, \delta(E_1' + E_2' - E)\, \mathrm{d}^3\boldsymbol{p}_1'\, \mathrm{d}^3\boldsymbol{p}_2'.\end{aligned}$$

The delta function fixes $\boldsymbol{p}'_1$ so that

$$\delta^4(p_\beta - p_\alpha)\,\mathrm{d}\beta \to \delta(E'_1 + E'_2 - E)\,\mathrm{d}^3\boldsymbol{p}'_2.$$

where $E'_1 = \sqrt{|\boldsymbol{p}'_1|^2 + m'^2_1}$ with $\boldsymbol{p}'_1 \equiv \boldsymbol{k} - \boldsymbol{p}'_2$. Moreover, because $|\boldsymbol{p}'_1|^2 = |\boldsymbol{k}|^2 + |\boldsymbol{p}'_2|^2 - 2\boldsymbol{k}\cdot\boldsymbol{p}'_2 \equiv k^2 + |\boldsymbol{p}'_2|^2 - 2k|\boldsymbol{p}'_2|\cos\theta$, where θ is defined to be the angle between vectors $\boldsymbol{k}$ and $\boldsymbol{p}'_2$ we can write the energy δ-function explicitly as

$$\delta(E'_1 + E'_2 - E)\,\mathrm{d}^3\boldsymbol{p}'_2 = \delta\left(\sqrt{k^2 + |\boldsymbol{p}'_2|^2 - 2k|\boldsymbol{p}'_2|\cos\theta + m'^2_1} + \sqrt{|\boldsymbol{p}'_2|^2 + m'^2_2} - E\right)|\boldsymbol{p}'_2|^2\mathrm{d}|\boldsymbol{p}'_2|\,\mathrm{d}\Omega.$$

The remaining δ-function can be rewritten using the identity

$$\delta(f(x)) = \sum_n \frac{\delta(x - x^*_n)}{f'(x^*_n)}$$

where $\{x^*_1, x^*_2, \cdots\}$ are the zeros of $f(x)$. We must therefore determine the roots of

$$\sqrt{k^2 + |\boldsymbol{p}'_2|^2 - 2k|\boldsymbol{p}'_2|\cos\theta + m'^2_1} + \sqrt{|\boldsymbol{p}'_2|^2 + m'^2_2} - E = 0.$$

Although there are two roots, only one is positive-definite. We write this solution as $|\boldsymbol{p}'_2| = k'$ where*

$$k' \equiv \frac{k\cos\theta(E^2 - m'^2_1 - k^2 + m'^2_2) + E\sqrt{(E^2 - m'^2_1 - m'^2_2 - k^2)^2 - 4\,m'^2_1 m'^2_2 - 4k^2 m'^2_2 \sin^2\theta}}{2(E^2 - k^2\cos^2\theta)}$$

$$E'_1 \equiv \sqrt{|\boldsymbol{k}' - \boldsymbol{k}|^2 + m'^2_1} = \sqrt{k'^2 + k^2 - 2kk'\cos\theta + m'^2_1}$$

$$E'_2 \equiv \sqrt{k'^2 + m'^2_2}.$$

The derivative (evaluated at $|\boldsymbol{p}'_2| = k'$) is

$$\frac{k'}{\sqrt{k'^2 + m'^2_2}} + \frac{k' - k\cos\theta}{\sqrt{k'^2 + k^2 - 2kk'\cos\theta + m'^2_1}} = \frac{k'}{E'_2} + \frac{k' - k\cos\theta}{E'_2}$$

and so

$$\delta(E'_1 + E'_2 - E)\,\mathrm{d}^3\boldsymbol{p}'_2 \to k'^2\left(\frac{k'}{E'_2} + \frac{k' - k\cos\theta}{E'_1}\right)^{-1}\mathrm{d}\Omega$$

which we can plug into (3.4.15) to obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = (2\pi)^4 \frac{E_2}{k} k'^2 \left(\frac{k'}{E_2'} + \frac{k' - k\cos\theta}{E_1'}\right)^{-1} |M_{\beta\alpha}|^2$$
$$= (2\pi)^4 \frac{k'}{k} \left(E_1' + E_2'(1 - \frac{k}{k'}\cos\theta)\right)^{-1} \frac{E_2}{E_1' E_2'} |M_{\beta\alpha}|^2$$

where we have used eqn. (3.4.17) to evaluate u_α in the laboratory frame as

$$u_\alpha = \frac{|\boldsymbol{p}_2|}{E_2} = \frac{k}{E - m_1}.$$

*As a safety check, note that in the special case $\boldsymbol{k} = 0$, which corresponds to the center of mass frame ($\boldsymbol{p}_1' + \boldsymbol{p}_2' = 0$) the equations below reduce to (3.4.24 – 3.4.26).

Problem 4

Derive the perturbation expansion (3.5.8) directly from the expansion (3.5.3) of old-fashioned perturbation theory.

Solution

To show (3.5.8) is equivalent to (3.5.3), we need to evaluate $S_{\beta\alpha}$ from (3.5.3):

$$S_{\beta\alpha} = (\Phi_\beta, S\Phi_\alpha)$$
$$= (\Phi_\beta, \Phi_\alpha) + (-i)\int_{-\infty}^{\infty} \mathrm{d}t_1\, (\Phi_\beta, V(t_1)\Phi_\alpha) + (-i)^2 \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2\, (\Phi_\beta, V(t_1)V(t_2)\Phi_\alpha) + ... \tag{3-4-1}$$

The term 0-th order in V is simply:

$$(\Phi_\beta, \Phi_\alpha) = \delta(\beta - \alpha) \tag{3-4-2}$$

To evaluate the 1-st order term, we first show:

$$(\Phi_\beta, V(t)\Phi_\alpha) = (\Phi_\beta, e^{iH_0 t} V e^{-iH_0 t}\Phi_\alpha) = (e^{-iH_0 t}\Phi_\beta, V e^{-iH_0 t}\Phi_\alpha) = e^{i(E_\beta - E_\alpha)t} V_{\beta\alpha} \tag{3-4-3}$$

So:

$$(-i)\int_{-\infty}^{\infty} \mathrm{d}t_1\, (\Phi_\beta, V(t_1)\Phi_\alpha) = (-i)\int_{-\infty}^{\infty} \mathrm{d}t_1\, e^{i(E_\beta - E_\alpha)t_1} V_{\beta\alpha} = (-2\pi i)\delta(E_\beta - E_\alpha)V_{\beta\alpha} \tag{3-4-4}$$

For the 2-nd order term, we can use the completeness of the Φ_α's to show:

$$(\Phi_\beta, V(t_1)V(t_2)\Phi_\alpha) = \int \mathrm{d}\gamma\, (\Phi_\beta, V(t_1)\Phi_\gamma)(\Phi_\gamma, V(t_2)\Phi_\alpha) \tag{3-4-5}$$

So:

$$
\begin{aligned}
&(-i)^2\int_{-\infty}^{\infty}\mathrm{d}t_1\int_{-\infty}^{t_1}\mathrm{d}t_2\,(\Phi_\beta, V(t_1)V(t_2)\Phi_\alpha)\\
=&(-i)^2\int\mathrm{d}\gamma\int_{-\infty}^{\infty}\mathrm{d}t_1\int_{-\infty}^{t_1}\mathrm{d}t_2\,e^{i(E_\beta-E_\gamma)t_1}e^{i(E_\gamma-E_\alpha)t_2}V_{\beta\gamma}V_{\gamma\alpha}
\end{aligned}
\tag{3-4-6}
$$

Now we do a change of integration variables in the form of:

$$
\tau = t_1 - t_2,\quad t_2' = t_2 \tag{3-4-7}
$$

Then the integral becomes:

$$
\begin{aligned}
&(-i)^2\int\mathrm{d}\gamma\int_0^{\infty}\mathrm{d}\tau\int_{-\infty}^{\infty}\mathrm{d}t_2\,e^{i(E_\beta-E_\gamma)(\tau+t_2)}e^{i(E_\gamma-E_\alpha)t_2}V_{\beta\gamma}V_{\gamma\alpha}\\
=&(-i)^2\int\mathrm{d}\gamma\int_0^{\infty}\mathrm{d}\tau\,2\pi\delta(E_\beta-E_\alpha)e^{i(E_\beta-E_\gamma)\tau}V_{\beta\gamma}V_{\gamma\alpha}\\
=&(-2\pi i)\delta(E_\beta-E_\alpha)\int\mathrm{d}\gamma\,(-i)\int_0^{\infty}\mathrm{d}\tau\,e^{i(E_\alpha-E_\gamma)\tau}V_{\beta\gamma}V_{\gamma\alpha}\\
=&(-2\pi i)\delta(E_\beta-E_\alpha)\int\mathrm{d}\gamma\,\frac{V_{\beta\gamma}V_{\gamma\alpha}}{E_\alpha-E_\gamma+i\epsilon}
\end{aligned}
\tag{3-4-8}
$$

where we used (3.5.9) in the last equality. To evaluate the higher order terms, we first use the completeness relation for Φ_α's (as in the case for the 2-nd order term) to expand $(\Phi_\beta, V(t_1)...V(t_n)\Phi_\alpha)$:

$$
(\Phi_\beta, V(t_1)...V(t_n)\Phi_\alpha) = \int\mathrm{d}\gamma_1\,...\int\mathrm{d}\gamma_{n-1}\,e^{i(E_\beta-E_{\gamma_1})t_1}...e^{i(E_{\gamma_{n-1}}-E_\alpha)t_n}V_{\beta\gamma_1}...V_{\gamma_{n-1}\alpha} \tag{3-4-9}
$$

And then evaluate the $t_1...t_n$ integral by doing a change of variables:

$$
\begin{aligned}
&\tau_i = t_i - t_{i+1}\quad\text{for}\quad i\le n-1\\
&t_n' = t_n
\end{aligned}
\tag{3-4-10}
$$

The procedure is similar to that for the 2-nd order term. Combining the results, we get:

$$
S_{\beta\alpha} = \delta(\beta-\alpha) - 2\pi i\delta(E_\beta-E_\alpha)\Big[V_{\beta\alpha} + \int\mathrm{d}\gamma\,\frac{V_{\beta\gamma}V_{\gamma\alpha}}{E_\alpha-E_\gamma+i\epsilon} + ...\Big] \tag{3-4-11}
$$

The term in the bracket is simply $T^+_{\beta\alpha}$. We can see that it is consistent with (3.5.3).

Chapter 4

Problem 1

Define generating functionals for the S-matrix and its connected part :

$$\begin{aligned} F[v] \equiv & 1+\sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{1}{N!M!}\int v^*(q_1')\cdots v^*(q_N')v(q_1)\cdots v(q_M) \\ & \times S_{q_1'\dots q_N',q_1\dots q_M}\mathrm{d}q_1'\cdots \mathrm{d}q_N'\,\mathrm{d}q_1\cdots \mathrm{d}q_M \\ F^C[v] \equiv & 1+\sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{1}{N!M!}\int v^*(q_1')\cdots v^*(q_N')v(q_1)\cdots v(q_M) \\ & \times S^C_{q_1'\dots q_N',q_1\dots q_M}\mathrm{d}q_1'\cdots \mathrm{d}q_N'\,\mathrm{d}q_1\cdots \mathrm{d}q_M. \end{aligned}$$

Derive a formula relating $F[v]$ and $F^C[v]$. (You may consider the purely bosonic case.)

Solution

We can modify (4.3.2) to write:

$$\begin{aligned} S_{\beta\alpha} &= \sum_{\text{PART}} S^C_{\beta_1\alpha_1}S^C_{\beta_2\alpha_2}\cdots \\ &= \sum_{n=1}^{\infty}\sum_{\substack{\text{PART}\\ \text{in n}}} S^C_{\beta_1\alpha_1}\dots S^C_{\beta_n\alpha_n} \end{aligned} \tag{4-1-1}$$

Here, PART in n is the sum over different partitions with n parts. (see (4.3.4), (4.3.5), (4.3.6)) Now we have:

$$\begin{aligned} & F[v] \\ = & 1+\sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{1}{N!M!}\int v^*(q_1')\dots v^*(q_N')v(q_1)\dots v(q_M) \\ & S_{q_1'\dots q_N',q_1\dots q_M}\,\mathrm{d}q_1'\dots \mathrm{d}q_N'\,\mathrm{d}q_1\dots \mathrm{d}q_M \\ = & 1+\sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{1}{N!M!}\int v^*(q_1')\dots v^*(q_N')v(q_1)\dots v(q_M) \\ & \sum_{n=1}^{\infty}\sum_{\substack{\text{PART}\\ \text{in n}}} S^C_{\beta_1\alpha_1}\dots S^C_{\beta_n\alpha_n}\,\mathrm{d}q_1'\dots \mathrm{d}q_N'\,\mathrm{d}q_1\dots \mathrm{d}q_M \end{aligned} \tag{4-1-2}$$

Here, we have expanded $S_{q_1'\dots q_N',q_1\dots q_M}$ in terms of $S^C_{\beta_1\alpha_1}\dots S^C_{\beta_n\alpha_n}$. The β_i's give a partition for $q_1'\dots q_N'$, and similarly for the α_i's. If the size of β_i is N_i and the size of α_i is M_i, then we can write the sum over partitioning as a sum over different sizes N_i and M_i, satisfying:

$$N_1+\ldots+N_n=N, \quad M_1+\ldots+M_n=M. \tag{4-1-3}$$

The number of partitionings of q_i''s and q_i's of the a particular size N_i and M_i is given by:

$$\frac{N!}{N_1!...N_n!}, \quad \frac{M!}{M_1!...M_n!}, \tag{4-1-4}$$

respectively. Since we're integrating over all the q_i''s and q_i's, the contribution from all terms of a particular partitioning size is the same, so we must add the above factors inside the summation of the N_i and M_i's.

Moreover, since we are interested in distinct partitionings, a partitioning $\beta_i\alpha_i$ should be treated as the same as $\beta_{k_i}\alpha_{k_i}$, where k_i is some permutation of $\{1,...n\}$. To avoid over counting, we should divided the sum by $n!$.

Continuing with our previous calculation, we have:

$$\begin{aligned}
&1+\sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{1}{N!M!}\int v^*(q_1')...v^*(q_N')v(q_1)...v(q_M)\\
&\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{N_1+...N_n=N}\ \sum_{M_1+...M_n=M}\frac{N!}{N_1!...N_n!}\frac{M!}{M_1!...M_n!}\\
&S^C_{q_1'...q_{N_1}',q_1...q_{M_1}}...S^C_{q_{N-N_n+1}'...q_N',q_{M-M_n+1}...q_M}\,\mathrm{d}q_1'\,...\,\mathrm{d}q_N'\,\mathrm{d}q_1\,...\,\mathrm{d}q_M\\
=&\,1+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{N=n}^{\infty}\sum_{M=n}^{\infty}\ \sum_{N_1+...N_n=N}\ \sum_{M_1+...M_n=M}\frac{1}{N_1!...N_n!}\frac{1}{M_1!...M_n!}\\
&\int v^*(q_1')...v^*(q_N')v(q_1)...v(q_M)S^C_{q_1'...q_{N_1}',q_1...q_{M_1}}...S^C_{q_{N-N_n+1}'...q_N',q_{M-M_n+1}...q_M}\,\mathrm{d}q_1'\,...\,\mathrm{d}q_N'\,\mathrm{d}q_1\,...\,\mathrm{d}q_M\\
=&\,1+\sum_{n=1}^{\infty}\frac{1}{n!}\Big(\prod_{k=1}^{n}\sum_{N_k=1}^{\infty}\Big)\Big(\prod_{k=1}^{n}\sum_{M_k=1}^{\infty}\Big)\frac{1}{N_1!...N_n!}\frac{1}{M_1!...M_n!}\\
&\int v^*(q_1')...v^*(q_N')v(q_1)...v(q_M)S^C_{q_1'...q_{N_1}',q_1...q_{M_1}}...S^C_{q_{N-N_n+1}'...q_N',q_{M-M_n+1}...q_M}\,\mathrm{d}q_1'\,...\,\mathrm{d}q_N'\,\mathrm{d}q_1\,...\,\mathrm{d}q_M\\
=&\,1+\sum_{n=1}^{\infty}\frac{1}{n!}\prod_{k=1}^{n}\sum_{N_k=1}^{\infty}\sum_{M_k=1}^{\infty}\frac{1}{N_k!M_k!}\\
&\int v^*(q_1')...v^*(q_{N_k}')v(q_1)...v(q_{M_k})S^C_{q_1'...q_{N_k}',q_1...q_{M_k}}\,\mathrm{d}q_1'\,...\,\mathrm{d}q_{N_k}'\,\mathrm{d}q_1\,...\,\mathrm{d}q_{M_k}\\
=&\,1+\sum_{n=1}^{\infty}\frac{1}{n!}(F^C[v])^n\\
=&\,\exp\big\{F^C[v]\big\}
\end{aligned}\tag{4-1-5}$$

So the desired formula is $F[v]=\exp\big\{F^C[v]\big\}$.

Problem 2

THIS IS A WORK IN PROGRESS.

Consider an interaction

$$V = g\int \mathrm{d}\boldsymbol{p}_1\,\mathrm{d}\boldsymbol{p}_2\,\mathrm{d}\boldsymbol{p}_3\,\mathrm{d}\boldsymbol{p}_4\,\delta^3(\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{p}_3-\boldsymbol{p}_4)\;a^\dagger(\boldsymbol{p}_1)a^\dagger(\boldsymbol{p}_2)a(\boldsymbol{p}_3)a(\boldsymbol{p}_4)$$

where g is a real constant and $a(\boldsymbol{p})$ is the annihilation operator of a spinless boson of mass $M > 0$. Use perturbation theory to calculate the S-matrix element for scattering of these particles in the center-of-mass frame to order g^2. What is the corresponding differential cross-section?

Solution

In general, the S-matrix elements can be computed from the connected parts S^C. The connected parts of the S-matrix can be calculated perturbatively as

$$S^C_{\beta\alpha} = \delta(\beta-\alpha) - i\int_{-\infty}^{\infty}\mathrm{d}t\,\langle\Phi_\beta|\,V(t)\,|\Phi_\alpha\rangle_C + \frac{(-i)^2}{2}\int \mathrm{d}t_1\,\mathrm{d}t_2\,\langle\Phi_\beta|\,T\{V(t_1)V(t_2)\}\,|\Phi_\alpha\rangle_C + \cdots \tag{4-2-1}$$

where $|\Phi_{\alpha,\beta}\rangle$ are free particle states and the subscript C denotes connected contributions only. Let's label the terms in eqn. (4-2-1) as $S^C_{\beta\alpha} = S^{C(0)}_{\beta\alpha} + S^{C(1)}_{\beta\alpha} + S^{C(2)}_{\beta\alpha} + \cdots$. First we focus on $S^{C(1)}_{\beta\alpha}$. Suppose the states $|\Phi_\alpha\rangle$ and $|\Phi_\beta\rangle$ consist of n and m particles respectively, e.g. if $n = 2$ and $m = 3$ we would have something like

$$|\Phi_\alpha\rangle = a^\dagger_{\boldsymbol{p}_A} a^\dagger_{\boldsymbol{p}_B}\,|\Phi_0\rangle\,, \qquad |\Phi_\beta\rangle = a^\dagger_{\boldsymbol{p}'_A} a^\dagger_{\boldsymbol{p}'_B} a^\dagger_{\boldsymbol{p}'_C}\,|\Phi_0\rangle$$

and the connected S-matrix element would be proportional to the vacuum expectation value $\langle\Psi_0|\,a_{\boldsymbol{p}'_C} a_{\boldsymbol{p}'_B} a_{\boldsymbol{p}'_A}\,a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2} a_{\boldsymbol{p}_3} a_{\boldsymbol{p}_4}\,a^\dagger_{\boldsymbol{p}_A} a^\dagger_{\boldsymbol{p}_B}\,|\Psi_0\rangle$, but this vanishes because if we use the creation-annihilation commutation relations to write the product of operators in normal order it is impossible to produce a term that is exclusively a product of δ-function factors – there will always be a creation (or annihilation) operator left over which will hit the vacuum on the left (or right), giving 0. Hence $S^{C(1)}_{\beta\alpha}$ vanishes for all $|\Psi_{\alpha,\beta}\rangle$ where $N_\alpha \neq N_\beta$. Furthermore, by similar reasoning we can determine that $S^{C(1)}_{\beta\alpha}$ vanishes for $n \to n$ scattering processes when $n > 2$ and when $n = 1$. Consequently we must only compute $S^{C(1)}_{\beta\alpha}$ in the case of

$2 \to 2$ scattering:

$$
\begin{aligned}
S^{C(1)}_{\beta\alpha} &= (-ig)\int_{-\infty}^{\infty} \mathrm{d}t\, e^{i(E_\beta - E_\alpha)} \int \prod_{i=1}^{4} \mathrm{d}^3\boldsymbol{p}_i\, \delta^3(\boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_3 - \boldsymbol{p}_4)\ \langle \Phi_\beta |\, a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2} a_3 a_4\, | \Phi_\alpha \rangle \\
&= (-ig)\, 2\pi\, \delta(E_\beta - E_\alpha)\ \langle \Phi_0 |\, a_{\boldsymbol{p}_{A'}} a_{\boldsymbol{p}_{B'}} \int \prod_{i=1}^{4} \mathrm{d}^3\boldsymbol{p}_i\ \delta^3(\boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_3 - \boldsymbol{p}_4)\ a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2} a_{\boldsymbol{p}_3} a_{\boldsymbol{p}_4} \\
&\quad \times a^\dagger_{\boldsymbol{p}_A} a^\dagger_{\boldsymbol{p}_B}\, | \Phi_0 \rangle\,. \\
&= 2\pi\, (-ig)\, \delta(E_\beta - E_\alpha) \int \prod_{i=1}^{4} \mathrm{d}^3\boldsymbol{p}_i\ \delta^3_{\boldsymbol{P}}\, \langle a_{\boldsymbol{p}_{A'}} a_{\boldsymbol{p}_{B'}}\ a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2} a_{\boldsymbol{p}_3} a_{\boldsymbol{p}_4} a^\dagger_{\boldsymbol{p}_A} a^\dagger_{\boldsymbol{p}_B} \rangle \\
&= \boxed{(2\pi)\, (-ig) \delta^4(p_\alpha - p_\beta)\, 4}
\end{aligned}
$$

where in the third equality we defined $\delta^3_{\boldsymbol{P}} \equiv \delta^3(\boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_3 - \boldsymbol{p}_4)$ and in the last equality we have defined $\delta^4(p_\alpha - p_\beta) \equiv \delta(E_\alpha - E_\beta)\, \delta^3(\boldsymbol{p}_{A'} + \boldsymbol{p}_{B'} - \boldsymbol{p}_A - \boldsymbol{p}_B)$.

For $\mathcal{O}(g^2)$ we can make similar arguments as for $\mathcal{O}(g)$ to conclude that the only nonzero $S^{C(2)}_{\beta\alpha}$ elements will come from $2 \to 2$ and $3 \to 3$ scattering. Proceeding with the calculation with general states $|\Phi_{\alpha,\beta}\rangle$ for now we have

$$
\begin{aligned}
&(-ig)^2 \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \int \prod_{i=1}^{4} \mathrm{d}^3\boldsymbol{p}_i \prod_{j=1}^{4} \mathrm{d}^3\boldsymbol{p}'_j\, \delta^3_{\boldsymbol{P}}\, \delta^3_{\boldsymbol{P'}} \\
&\qquad \times \langle \Phi_\beta |\, e^{iH_0 t_1} a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2} a_{\boldsymbol{p}_3} a_{\boldsymbol{p}_4} e^{-iH_0 t_1} e^{iH_0 t_2}\ a^\dagger_{\boldsymbol{p}'_1} a^\dagger_{\boldsymbol{p}'_2} a_{\boldsymbol{p}'_3} a_{\boldsymbol{p}'_4} e^{-iH_0 t_2}\, | \Phi_\alpha \rangle
\end{aligned}
$$

Using the fact that $|\Phi_{\alpha,\beta}\rangle$ are free particle eigenstates this expression becomes

$$
\begin{aligned}
&(-ig)^2 \int \prod_{i=1}^{4} \mathrm{d}^3\boldsymbol{p}_i \prod_{j=1}^{4} \mathrm{d}^3\boldsymbol{p}'_j\, \delta^3_{\boldsymbol{P}}\, \delta^3_{\boldsymbol{P'}} \int_{-\infty}^{\infty} \mathrm{d}t_1 e^{i(E_\beta - E_\alpha - E'_1 - E'_2 + E'_3 + E'_4)t_1} \int_{-\infty}^{t_1} \mathrm{d}t_2\, e^{i(E'_1 + E'_2 - E'_3 - E'_4)t_2} \\
&\qquad \times \langle \Phi_\beta |\, a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2} a_{\boldsymbol{p}_3} a_{\boldsymbol{p}_4}\ a^\dagger_{\boldsymbol{p}'_1} a^\dagger_{\boldsymbol{p}'_2} a_{\boldsymbol{p}'_3} a_{\boldsymbol{p}'_4}\, | \Phi_\alpha \rangle
\end{aligned}
$$

Explicitly evaluating the integral over t_2 by inserting a convergence factor $e^{\varepsilon t_2}$ with $\varepsilon = 0+$ yields

$$
\begin{aligned}
&2\pi\, (-ig)^2\, \delta(E_\beta - E_\alpha) \int \prod_{i=1}^{4} \mathrm{d}^3\boldsymbol{p}_i \prod_{j=1}^{4} \mathrm{d}^3\boldsymbol{p}'_j\, \delta^3_{\boldsymbol{P}}\, \delta^3_{\boldsymbol{P'}}\, \frac{1}{i(E'_1 + E'_2 - E'_3 - E'_4)} \\
&\qquad \times \langle \Phi_\beta |\, a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2} a_{\boldsymbol{p}_3} a_{\boldsymbol{p}_4}\ a^\dagger_{\boldsymbol{p}'_1} a^\dagger_{\boldsymbol{p}'_2} a_{\boldsymbol{p}'_3} a_{\boldsymbol{p}'_4}\, | \Phi_\alpha \rangle
\end{aligned}
$$

the only states that will yield nonzero S-matrix elements are $n \to n$ particle scattering events. We will now consider two and three particle scattering separately.

<u>$2 \to 2$ scattering:</u> For two particle scattering when we commute the annihilation operators on the left all the way to the right we will produce nonzero terms in a series with terms

where left creation operators are paired with right annihilation operators like

$$\langle a_{\boldsymbol{p}'_A} a_{\boldsymbol{p}'_B} \, a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2} a_{\boldsymbol{p}_3} a_{\boldsymbol{p}_4} a^\dagger_{\boldsymbol{p}'_1} a^\dagger_{\boldsymbol{p}'_2} a_{\boldsymbol{p}'_3} a_{\boldsymbol{p}'_4} \, a^\dagger_{\boldsymbol{p}_A} a^\dagger_{\boldsymbol{p}_B} \rangle. \tag{4-2-2}$$

The overbar notation here simply denotes delta functions resulting from swapping the paired operators, e.g. the above expression leads to

$$\delta^3(\boldsymbol{p}_1 - \boldsymbol{p}'_A)\delta^3(\boldsymbol{p}_2 - \boldsymbol{p}'_B)\delta^3(\boldsymbol{p}'_1 - \boldsymbol{p}_3)\delta^3(\boldsymbol{p}'_2 - \boldsymbol{p}_4)\delta^3(\boldsymbol{p}_A - \boldsymbol{p}'_3)\delta^3(\boldsymbol{p}_B - \boldsymbol{p}'_4)$$

under the integral this term would yield

$$\int \prod_{i=1}^{4} \mathrm{d}^3\boldsymbol{p}_i \prod_{j=1}^{4} \mathrm{d}^3\boldsymbol{p}'_j \, \frac{1}{i(E'_1 + E'_2 - E'_3 - E'_4)} \, \delta^3(\boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_3 - \boldsymbol{p}_4) \, \delta^3(\boldsymbol{p}'_1 + \boldsymbol{p}'_2 - \boldsymbol{p}'_3 - \boldsymbol{p}'_4)$$
$$\times \, \delta^3(\boldsymbol{p}_1 - \boldsymbol{p}'_A)\delta^3(\boldsymbol{p}_2 - \boldsymbol{p}'_B)\delta^3(\boldsymbol{p}'_1 - \boldsymbol{p}_3)\delta^3(\boldsymbol{p}'_2 - \boldsymbol{p}_4)\delta^3(\boldsymbol{p}_A - \boldsymbol{p}'_3)\delta^3(\boldsymbol{p}_B - \boldsymbol{p}'_4)$$

$$= \delta^3(\boldsymbol{p}'_A + \boldsymbol{p}'_B - \boldsymbol{p}_A - \boldsymbol{p}_B) \, \int \mathrm{d}^3\boldsymbol{p}'_1 \frac{1}{i(E_{\boldsymbol{p}'_1} + E_{\boldsymbol{p}_A + \boldsymbol{p}_B - \boldsymbol{p}'_1} - E_{\boldsymbol{p}_A} - E_{\boldsymbol{p}_B})}.$$

In the final equality we see that one of the momenta ($\boldsymbol{p}'_1$) is not fixed. So we choose to relabel it as $\boldsymbol{\ell}$:

$$= \delta^3(\boldsymbol{p}'_A + \boldsymbol{p}'_B - \boldsymbol{p}_A - \boldsymbol{p}_B) \, \int \mathrm{d}^3\boldsymbol{\ell} \frac{1}{i(E_{\boldsymbol{\ell}} + E_{\boldsymbol{p}_A + \boldsymbol{p}_B - \boldsymbol{\ell}} - E_{\boldsymbol{p}_A} - E_{\boldsymbol{p}_B})}.$$

With a little thought you can convince yourself that there are 4! pairings which are equivalent to (4-2-2) e.g.

$$\langle a_{\boldsymbol{p}'_A} a_{\boldsymbol{p}'_B} \, a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2} a_{\boldsymbol{p}_3} a_{\boldsymbol{p}_4} a^\dagger_{\boldsymbol{p}'_1} a^\dagger_{\boldsymbol{p}'_2} a_{\boldsymbol{p}'_3} a_{\boldsymbol{p}'_4} \, a^\dagger_{\boldsymbol{p}_A} a^\dagger_{\boldsymbol{p}_B} \rangle.$$

In fact all contributions to $2 \to 2$ scattering at this order will give the same contribution. Thus, for $2 \to 2$ scattering we have

$$\boxed{S^{C(2)}_{\beta\alpha} = 2\pi \, 4! \, (-ig)^2 \, \delta^4(p_\beta - p_\alpha) \, \int \mathrm{d}^3\boldsymbol{\ell} \frac{1}{i(E_{\boldsymbol{\ell}} + E_{\boldsymbol{p}_A + \boldsymbol{p}_B - \boldsymbol{\ell}} - E_{\boldsymbol{p}_A} - E_{\boldsymbol{p}_B})}}\,.$$

<u>$3 \to 3$ scattering:</u>

$$\langle a_{\boldsymbol{p}'_A} a_{\boldsymbol{p}'_B} a_{\boldsymbol{p}'_C} \, a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2} a_{\boldsymbol{p}_3} a_{\boldsymbol{p}_4} a^\dagger_{\boldsymbol{p}'_1} a^\dagger_{\boldsymbol{p}'_2} a_{\boldsymbol{p}'_3} a_{\boldsymbol{p}'_4} \, a^\dagger_{\boldsymbol{p}_A} a^\dagger_{\boldsymbol{p}_B} a^\dagger_{\boldsymbol{p}_C} \rangle.$$

Chapter 5

Problem 1

Show that if the zero-momentum coefficient functions satisfy the conditions (5.1.23) and (5.1.24), then the coefficient functions (5.1.21) and (5.1.22) for arbitrary momentum satisfy the defining conditions Eqs. (5.1.19) and (5.1.20).

Solution

We only complete this problem for the u coefficient functions because the calculation for v is almost identical. From (5.1.23) we have*

$$\sum_{\sigma'} u_\ell(0,\sigma')\, D^{(j)}_{\sigma'\sigma}(R) = \sum_{\ell'} D_{\ell\ell'}(R)\, u_{\ell'}(0,\sigma) \tag{5-1-1}$$

and from (5.1.21) we have

$$u_\ell(\boldsymbol{p},\sigma) = (m/q^0)^{1/2} \sum_{\ell'} D_{\ell\ell'}(L(p))\, u_{\ell'}(0,\sigma)$$

where R is an arbitrary 3-dimensional rotation (that is, an element of $SO(3)$). We wish to construct a relationship between $u_\ell(\boldsymbol{p},\sigma)$ and its Lorentz-transformed counterpart $u_\ell(\boldsymbol{p}_\Lambda,\sigma)$. To do this we can invert (5.1.21) to write $u_\ell(0,\sigma)$ in terms of $u_\ell(\boldsymbol{p},\sigma)$:

$$u_{\ell'}(0,\sigma) = (p^0/m)^{1/2} \sum_{\ell''} D_{\ell'\ell''}\left(L_p^{-1}\right) u_{\ell''}(\boldsymbol{p},\sigma). \tag{5-1-2}$$

Of course, this relationship holds for any momentum $\boldsymbol{p}$. In particular, we can replace $\boldsymbol{p}$ with $\boldsymbol{p}_\Lambda$ where Λ is an arbitrary proper orthochronous Lorentz transformation and $\boldsymbol{p}_\Lambda$ is the 3-momentum part of Λp:

$$u_\ell(0,\sigma) = ((\Lambda p)^0/m)^{1/2} \sum_{\ell'} D_{\ell\ell'}\left(L_{\Lambda p}^{-1}\right) u_{\ell'}(\boldsymbol{p}_\Lambda,\sigma). \tag{5-1-3}$$

We can now substitute (5-1-2) in the rhs of (5-1-1) and 5-1-3) into the lhs of (5-1-1) to obtain

$$\begin{aligned}\sum_{\sigma'}\sum_{\ell'} D_{\ell\ell'}\left(L_{\Lambda p}^{-1}\right) u_{\ell'}(\boldsymbol{p}_\Lambda,\sigma) D^{(j)}_{\sigma'\sigma}(R) &= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{\ell'\ell''} D_{\ell\ell'}(R) D_{\ell'\ell''}\left(L_p^{-1}\right) u_{\ell''}(\boldsymbol{p},\sigma) \\ &= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{\ell'} D_{\ell\ell'}(R\, L_p^{-1}) u_{\ell'}(\boldsymbol{p},\sigma).\end{aligned}$$

We can apply $D_{\ell\ell'}(L_{\Lambda p})$ to both sides of the equation to get rid of the sum over ℓ' that appears on the lhs. This yields

$$\sum_{\sigma'} u_\ell(\boldsymbol{p}_\Lambda, \sigma) D^{(j)}_{\sigma'\sigma}(R) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{\ell'} D_{\ell\ell'}(L_{\Lambda p} R\, L_p^{-1}) u_{\ell'}(\boldsymbol{p}, \sigma).$$

Now we can use the fact that the little group for massive particles is $SO(3)$ and choose the rotation R to be the Wigner rotation $W(\Lambda, p) \equiv L^{-1}_{\Lambda p} \Lambda L_p$. This immediately gives

$$\begin{aligned}
\sum_{\sigma'} u_\ell(\boldsymbol{p}_\Lambda, \sigma) D^{(j)}_{\sigma'\sigma}(W(\Lambda, p)) &= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{\ell'} D_{\ell\ell'}(L_{\Lambda p} W(\Lambda, p) L_p^{-1}) u_{\ell'}(\boldsymbol{p}, \sigma) \\
&= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{\ell'} D_{\ell\ell'}(L_{\Lambda p}(L^{-1}_{\Lambda p}\, \Lambda\, L_p) L_p^{-1}) u_{\ell'}(\boldsymbol{p}, \sigma) \\
&= \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{\ell'} D_{\ell\ell'}(\Lambda) u_{\ell'}(\boldsymbol{p}, \sigma).
\end{aligned}$$

Hence,

$$\sum_{\sigma'} u_\ell(\boldsymbol{p}_\Lambda, \sigma) D^{(j)}_{\sigma'\sigma}(W(\Lambda, p)) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{\ell'} D_{\ell\ell'}(\Lambda) u_{\ell'}(\boldsymbol{p}, \sigma)$$

which exactly matches (5.1.19), as we were trying to show.

*In the following we suppress all particle species labels n.

Problem 4

> Show that the fields for a massless particle of spin j of type $(A, A + j)$ or $(B + j, B)$ are the $2A$th or $2B$th derivatives of fields of type $(0, j)$ or $(j, 0)$, respectively.

Solution

A type $(A, A + j)$ field for a massless particle of spin j' satisfy:

$$|A - (A + j)| \leq j' \leq 2A + j$$

We see that j' can go from j to $2A + j$. For each j', we know from (5.9.41) that the field can be formed only from annihilation operators of helicity σ, where

$$\sigma = (A + j) - A = j.$$

The total number of degrees of freedom here is:

$$(2A + j) - j + 1 = 2A + 1$$

Now consider the traceless $2A$-th derivatives of a massless type $(0, j)$ field ϕ_σ:

$$\{\partial_{\mu_1} \dots \partial_{\mu_{2A}}\}\phi_\sigma$$

This object tranforms with a representation $(A, A) \otimes (0, j)$, where the (A, A) representation comes from the traceless derivatives and the $(0, j)$ is the transformation of the field. Again, since σ is fixed by (5.9.41), for each $0 \leq j \leq 2A$ there is 1 degree of freedom, and there is 1 degree of freedom for $(0, j)$. The total number of degree of freedom is:

$$(2A + 1) \times 1 = 2A + 1$$

Since we have obtained the same number of dof for a general (and unique) type $(A, A + j)$ massless field, we know that all such fields must be formed from traceless derivatives of type $(0, j)$ field. The argument is similar for type $(B + j, B)$ fields.

Problem 5

> Work out the transformation properties of fields of transformation type $(j, 0) + (0, j)$ for massless particles of helicity $\pm j$ under the inversions P, C, T.

Solution

For P, we can use result (5.7.43):

$$P\psi^{AB}_{ab}(x)P^{-1} = \eta^*(-1)^{A+B-j}\psi^{BA}_{ba}(-\mathbf{x}, x^0)$$

The $(j, 0)$ component of the field transforms like:

$$P\psi^{j0}_{ab}(x)P^{-1} = \eta^*\psi^{0j}_{ba}(-\mathbf{x}, x^0)$$

Since $\sigma = 0 - j = -j = a + b$, the only values for a and b are:

$$a = -j, \quad b = 0.$$

So the simplified transformation is:

$$P\psi^{j0}_{-j,0}(x)P^{-1} = \eta^*\psi^{0j}_{0,-j}(-\mathbf{x}, x^0)$$

Similarly, the $(0, j)$ component transforms like:

$$P\psi^{0j}_{0,j}(x)P^{-1} = \eta^*\psi^{j0}_{j,0}(-\mathbf{x}, x^0)$$

Note that the $(j, 0)$ component is transformed into the $(0, j)$ component, and vice versa. The full transformation is thus:

$$P\psi(x)P^{-1} = \eta^*\beta\psi(-\mathbf{x}, 0)$$

where

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

switches the two components.

Similarly, for C we use (5.7.52):

$$\begin{aligned} C\psi^{AB}_{ab}(x)C^{-1} =& \xi^*(-1)^{-2A-a-b-j}\psi^{BA\dagger}_{-b-a}(x) \\ C\psi^{j0}_{-j0}(x)C^{-1} =& \xi^*(-1)^{-2j-(-j)-0-j}\psi^{0j\dagger}_{0,j}(x) = \xi^*(-1)^{2j}\psi^{0j\dagger}_{0,j}(x) \\ C\psi^{0j}_{0j}(x)C^{-1} =& \xi^*(-1)^{-0-j-j}\psi^{j0\dagger}_{-j,0}(x) = \xi^*(-1)^{2j}\psi^{j0\dagger}_{-j,0}(x) \end{aligned}$$

For T, we use (5.7.60):

$$\begin{aligned} T\psi^{AB}_{ab}(x)T^{-1} =& (-1)^{a+b+A+B-2j}\zeta^*\psi^{AB}_{-a-b}(\mathbf{x}, -x^0) \\ T\psi^{j0}_{-j,0}(x)T^{-1} =& (-1)^{-j+0+j+0-2j}\zeta^*\psi^{j0}_{j,0}(\mathbf{x}, -x^0) = (-1)^{2j}\zeta^*\psi^{j0}_{j,0}(\mathbf{x}, -x^0) \\ T\psi^{0j}_{0,j}(x)T^{-1} =& (-1)^{0+j+0+j-2j}\zeta^*\psi^{0j}_{0,-j}(\mathbf{x}, -x^0) = \zeta^*\psi^{0j}_{0,-j}(\mathbf{x}, -x^0) \end{aligned}$$

Chapter 6

Problem 2

Consider a theory involving a neutral scalar field $\phi(x)$ for a boson B and a complex Dirac field $\psi(x)$ for a fermion F, with interaction (in the interaction picture) $V = ig\int \mathrm{d}^3x\, \bar{\psi}(x)\gamma_5\psi(x)\phi(x)$. Draw all the connected order-g^2 Feynman diagrams and calculate the corresponding S-matrix elements for the processes $F^c + B \to F^c + B$, $F + F^c \to F + F^c$, and $F^c + F \to B + B$ (where F^c is the antiparticle of F). Do all integrals.

Solution

Case $F^c + B \to F^c + B$ The matrix element is given below. See Figure 2 for the diagram.

$$
\begin{aligned}
&S_{p_3p_4,p_1p_2} \\
=&(2\pi)^{-6}(-i^2g)^2(2\pi)^8\frac{-i}{(2\pi)^4}\delta^4(p_1+p_2-p_3-p_4)(4E_2E_4)^{-1/2} \\
&\left[\bar{v}(p_1)\gamma_5\frac{-i\gamma_\mu(-p_1^\mu-p_2^\mu)+m_\psi}{(p_1+p_2)^2+m_\psi^2-i\epsilon}\gamma_5 v(p_3)+\bar{v}(p_1)\gamma_5\frac{-i\gamma_\mu(-p_1^\mu+p_4^\mu)+m_\psi}{(p_1-p_4)^2+m_\psi^2-i\epsilon}\gamma_5 v(p_3)\right]
\end{aligned}
$$

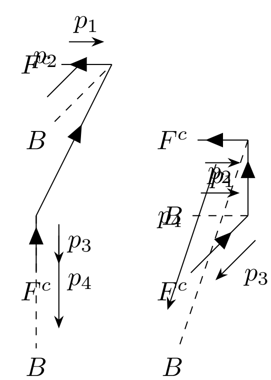

Figure 2: $F^c + B \to F^c + B$

Case $F + F^c \to F + F^c$ The matrix element is given below. See Figure 3 for the diagram.

$$
\begin{aligned}
&S_{p_3p_4,p_1p_2} \\
=&(2\pi)^{-6}(-i^2g)^2(2\pi)^8\frac{-i}{(2\pi)^4}\delta^4(p_1+p_2-p_3-p_4) \\
&\left[\bar{v}(p_2)\gamma_5 u(p_1)\frac{1}{(p_1+p_2)^2+m_\phi^2-i\epsilon}\bar{u}(p_3)\gamma_5 v(p_4)+\bar{u}(p_4)\gamma_5 u(p_1)\frac{1}{(p_1-p_4)^2+m_\phi^2-i\epsilon}\bar{v}(p_2)\gamma_5 v(p_3)\right]
\end{aligned}
$$

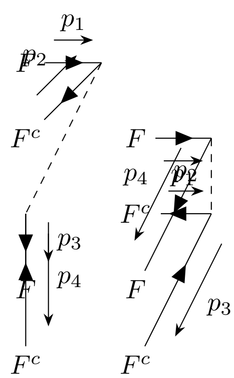

Figure 3: $F + F^c \to F + F^c$

Case $F^c + F \to B + B$ The matrix element is given below. See Figure 4 for the diagram.

$$
\begin{aligned}
&S_{p_3p_4,p_1p_2}\\
=&(2\pi)^{-6}(-i^2g)^2(2\pi)^8\frac{-i}{(2\pi)^4}\delta^4(p_1+p_2-p_3-p_4)(4E_3E_4)^{-1/2}\\
&\left[\bar{v}(p_1)\gamma_5\frac{-i\gamma_\mu(p_2^\mu-p_4^\mu)+m_\psi}{(p_2-p_4)^2+m_\psi^2-i\epsilon}\gamma_5u(p_2)+\bar{v}(p_1)\gamma_5\frac{-i\gamma_\mu(p_2^\mu-p_3^\mu)+m_\psi}{(p_2-p_3)^2+m_\psi^2-i\epsilon}\gamma_5u(p_2)\right]
\end{aligned}
$$

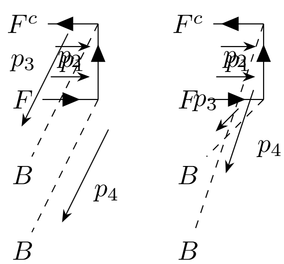

Figure 4: $F^c + F \to B + B$

Problem 4

What is the contribution in Feynman diagrams from the contraction of the derivative of the derivative $\partial_\mu\psi_\ell(x)$ of a Dirac field with the adjoint $\psi_m^\dagger(y)$ of the field?

Solution

Contributions to Feynman diagrams come from commuting annihilation operators on the left past creation operators on the right in time-ordered products $T\{\partial_\mu\psi_\ell(x)\psi_m^\dagger(y)\}$. We denote this contribution by $-i(\Delta_{\ell m})_\mu(x,y)$. From (5.5.34) the Dirac field is given by

$$\psi_\ell(x) = \psi_\ell^+(x) + \psi_\ell^{-c}(x),$$

where

$$\psi_\ell^+(x) \equiv (2\pi)^{-3/2} \sum_\sigma \int \mathrm{d}^3p \; u_\ell(\mathbf{p},\sigma)\, a(\mathbf{p},\sigma)\, e^{ip\cdot x},$$
$$\psi_\ell^{-c}(x) \equiv (2\pi)^{-3/2} \sum_\sigma \int \mathrm{d}^3p \; v_\ell(\mathbf{p},\sigma)\, b^\dagger(\mathbf{p},\sigma)\, e^{-ip\cdot x}.$$

The discussion in chapter 6.1 can be applied to derivatives of fields since* "from our point of view, the derivative of a field (6.1.3) is just another field described by (6.1.3), with different u_ℓ and v_ℓ" This means that the contribution in Feynman diagrams is given by (6.1.14) as

$$-i(\Delta_{\ell m})_\mu(x,y) = \theta(x-y)\left\{\partial_\mu\psi_\ell^+(x), \psi_m^{+\dagger}(y)\right\} - \theta(y-x)\left\{\psi_m^{-c\dagger}(y), \partial_\mu\psi_\ell^{-c}(x)\right\}$$

For $\left\{\partial_\mu\psi_\ell^+(x), \psi_m^{+\dagger}(y)\right\}$ we have

$$\begin{aligned}
\left\{\partial_\mu\psi_\ell^+(x), \psi_m^{+\dagger}(y)\right\} &= \partial_\mu \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left[\sum_\sigma u_\ell(\mathbf{p},\sigma)u_m^*(\mathbf{p},\sigma)\right] e^{ip\cdot(x-y)} \\
&= \partial_\mu \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2p^0}\left[(-i\gamma^\nu p_\nu + m)\beta\right]_{\ell m} e^{ip\cdot(x-y)} \\
&= \partial_\mu \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2p^0}\left[(-\gamma^\nu \partial_\nu + m)\beta\right]_{\ell m} e^{ip\cdot(x-y)} \\
&= \left[(-\gamma^\nu \partial_\nu + m)\beta\right]_{\ell m} \partial_\mu \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2p^0} e^{ip\cdot(x-y)} \\
&= \left[(-\gamma^\nu \partial_\nu + m)\beta\right]_{\ell m} \partial_\mu \Delta_+(x-y)
\end{aligned}$$

Similarly, for $\left\{\psi_m^{-c\dagger}(y), \partial_\mu\psi_\ell^{-c}(x)\right\}$ we have

$$\begin{aligned}
\left\{\psi_m^{-c\dagger}(y), \partial_\mu\psi_\ell^{-c}(x)\right\} &= \partial_\mu \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left[\sum_\sigma v_\ell(\mathbf{p},\sigma)v_m^*(\mathbf{p},\sigma)\right] e^{ip\cdot(y-x)} \\
&= -\left[(-\gamma^\nu \partial_\nu + m)\beta\right]_{\ell m} \partial_\mu \Delta_+(y-x)
\end{aligned}$$

So

$$\boxed{-i(\Delta_{\ell m})_\mu(x,y) = \left[(-\gamma^\nu \partial_\nu + m)\beta\right]_{\ell m} \partial_\mu\left[-i\Delta_F(x-y)\right]}$$

where $\Delta_F(x-y)$ is the *Feynman propagator* defined by (6.2.13).

Moreover, if we define

$$(P_{\ell m})_\mu(q) \equiv (iq_\mu)\left[(-i\gamma^\nu q_\nu + m)\beta\right]_{\ell m}$$

then using (6.3.3) we may immediately write the momentum space Feynman rule as

$$\boxed{-i(\Delta_{\ell m})_\mu(q) = \frac{-i}{(2\pi)^4}\frac{(iq_\mu)\left[(-i\gamma^\nu q_\nu + m)\beta\right]_{\ell m}}{q^2+m^2-i\varepsilon}}\ .$$

*quote from page 260.

Chapter 7

Problem 1

Consider the theory of a set of real scalar fields Φ^n, with Lagrangian density $\mathscr{L} = -\frac{1}{2}\sum_{mn}\partial_\mu\Phi^n\partial^\mu\Phi^m f_{nm}(\Phi)$, where $f_{nm}(\Phi)$ is an arbitrary real matrix function of the field. (This is called the *non-linear σ-model.*) Carry out the canonical quantization of this theory. Derive the interaction $V[\phi,\dot{\phi}(t)]$ in the interaction picture.

Solution

The Lagrangian density $\mathscr{L}$ may be rewritten as

$$\mathscr{L} = \frac{1}{2}\sum_{mn}\dot{\Phi}^n\dot{\Phi}^m f_{nm}(\Phi) - \frac{1}{2}\sum_{nm}(\vec{\nabla}\Phi^n)\cdot(\vec{\nabla}\Phi^m)f_{nm}(\Phi). \tag{7-1-1}$$

Therefore the conjugate momenta $\Pi_m \equiv \delta L/\delta\dot{\Phi}^m = \partial\mathscr{L}/\partial\dot{\Phi}^m$ (7.2.8) are given by

$$\Pi_n = \frac{\partial\mathscr{L}}{\partial\dot{\Phi}^n} = \sum_m \frac{1}{2}\dot{\Phi}^m\left[f_{nm}(\Phi) + f_{mn}(\Phi)\right] \tag{7-1-2}$$

We may take f to be symmetric since even if it wasn't we would be able to symmetrize it without changing the value of $\sum_{nm}\partial_\mu\Phi^n\partial^\mu\Phi^m f_{nm}(\Phi)$ (see footnote*.) We will also assume that f is real so that the reality of $\mathscr{L}$ is guaranteed. This means that we may also take f to be invertible (most matrices are anyway†) (7-1-2) can therefore be rewritten in matrix form as

$$\dot{\Phi} = f^{-1}\Pi. \tag{7-1-3}$$

The Hamiltonian is then simply

$$\begin{aligned}
H &= \int d^3x\left[\sum_n \Pi_n\dot{\Phi}_n - \mathscr{L}\right] \\
&= \int d^3x\left[\Pi^\top f^{-1}\,\Pi - \frac{1}{2}[f^{-1}\Pi]^\top f\,[f^{-1}\Pi] + \sum_{nm}\frac{1}{2}(\vec{\nabla}\Phi^n)\cdot(\vec{\nabla}\Phi^m)\,f_{nm}\right] \\
&= \int d^3x\sum_{nm}\left[\frac{1}{2}\Pi^\top f^{-1}\Pi + \sum_{nm}\frac{1}{2}(\vec{\nabla}\Phi^n)\cdot(\vec{\nabla}\Phi^m)\,f_{nm}\right] \\
&= \int d^3x\sum_{nm}\left[\frac{1}{2}\Pi_n[f^{-1}(\Phi)]_{nm}\Pi_m + \frac{1}{2}(\vec{\nabla}\Phi^n)\cdot(\vec{\nabla}\Phi^m)\,f_{nm}(\Phi)\right].
\end{aligned}$$

We now impose the canonical commutation relations in the Heisenberg picture (7.1.30-32).

$$[\Phi^n(\mathbf{x},t),\Pi_m(\mathbf{y},t)] = i\delta^3(\mathbf{x}-\mathbf{y})\delta_{nm}$$
$$[\Phi^n(\mathbf{x},t),\Phi^m(\mathbf{y},t)] = [\Pi_n(\mathbf{x},t),\Pi_m(\mathbf{y},t)] = 0$$

At the *picture-matching time* t_0 the Heisenberg operators Φ^n and Π_m are equal to their interaction picture counterparts ϕ^n and π_m. Since H is time-independent we can write it at $t = t_0$ by replacing Heisenberg operators with interaction-picture operators:

$$H = \int \mathrm{d}^3x \sum_{nm} \left[\frac{1}{2}\pi_n \, [f^{-1}(\phi)]_{nm} \, \pi_m + \frac{1}{2}(\vec{\nabla}\phi^n) \cdot (\vec{\nabla}\phi^m) \, f_{nm}(\phi) \right]. \tag{7-1-4}$$

Now we want to identify a suitable free Hamiltonian H_0 and an interaction $V[\phi, \dot{\phi}]$. Recall that for a single free field ϕ, the Hamiltonian would be given by

$$H_0 = \int \mathrm{d}^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi) \cdot (\vec{\nabla}\phi) + \frac{1}{2}m^2\phi^2 \right).$$

This motivates us to define matrices g and h satisfying $f \equiv 1 + g$ and $f^{-1} \equiv 1 + h$. This requires that g and h commute and also that $g + h + gh = 0$. With these definitions we can write the Hamiltonian as

$$\underbrace{\sum_i \int \mathrm{d}^3x \left(\frac{1}{2}\pi_i^2 + \frac{1}{2}(\vec{\nabla}\phi^i) \cdot (\vec{\nabla}\phi^i) \right)}_{H_0} + \underbrace{\int \mathrm{d}^3x \sum_{nm} \frac{1}{2} \Big([h(1+g)]_{nm} \, \dot{\phi^n} \dot{\phi^m} + g_{nm} (\vec{\nabla}\phi^n) \cdot (\vec{\nabla}\phi^m) \Big)}_{V[\phi, \dot{\phi}]}.$$

The first term is just a sum over individual free-field Hamiltonians for massless scalar fields ϕ^i. Therefore the remaining terms can be bundled into the interaction $V[\phi, \dot{\phi}]$.

---

*This statement is valid for any quadratic form $\sum_{ij} p_i p_j A^{ij}(x)$ so long as the components of p commute with each other (which is always the case for us.) One issue we might run into is if $[x, p] \neq 0$, which is exactly what we do when we canonically quantize. However, we only impose canonical commutation relations after computing the Hamiltonian so until then we don't have to pay them any attention.

To illustrate how symmetrisation works here's a simple example. Consider a 2×2 matrix A defined by

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}.$$

Further define $\bar{A} = (A + A^\top)/2$ so that

$$\bar{A} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}.$$

Then for $p = (p_1, p_2)^\top$

$$\begin{aligned} \sum_{ij} p_i p_j A^{ij} &= p_1^2 + 3p_1p_2 + 1p_2p_1 + 2p_2^2 &&= p_1^2 + 4p_1p_2 + 2p_2^2 \\ \sum_{ij} p_i p_j \bar{A}^{ij} &= p_1^2 + 2p_1p_2 + 2p_2p_1 + 2p_2^2 &&= p_1^2 + 4p_1p_2 + 2p_2^2. \end{aligned}$$

†A better argument is that if f wasn't invertible, then one of its eigenvalues would vanish, which would mean that some of the fields in Φ would not be independent and we could redefine the fields to get rid of the extraneous degrees of freedom. On the other hand then why not take f diagonal in the first place? I don't know.

Problem 3

In the theory described in Problem 2, suppose that the Lagrangian density is invariant under a global infinitesimal symmetry $\delta\Phi^n = i\epsilon \sum_m t^n{}_m \Phi^m$, $\delta\Psi^i = i\epsilon \sum_j \tau^i{}_j \Psi^j$. Derive an explicit expression for the conserved current associated with this symmetry.

Solution

The Lagrangian density is given by:

$$\mathcal{L} = \mathcal{L}_0(\Phi^n, \Psi^i, \partial_\mu \Phi^n, \partial_\mu \Psi^i) + \mathcal{L}_1(\Phi^n, \Psi^i).$$

This Lagrangian density is invariant under infinitesimal transformation:

$$\delta\Phi^n = i\epsilon t^n{}_m \Phi^m, \quad \delta\Psi^i = i\epsilon \tau^i{}_j \Psi^j.$$

Promoting ϵ to be a function of spacetime, $\mathcal{L}_0$ transforms as:

$$\delta\mathcal{L}_0 = \frac{\partial \mathcal{L}_0}{\partial \Phi^n} i\epsilon t^n{}_m \Phi^m + \frac{\partial \mathcal{L}_0}{\partial \Psi^i} i\epsilon \tau^i{}_j \Psi^j + \frac{\partial \mathcal{L}_0}{\partial \partial_\mu \Phi^n} i t^n{}_m \partial_\mu(\epsilon(x)\Phi^m(x)) + \frac{\partial \mathcal{L}_0}{\partial \partial_\mu \Psi^i} i\tau^i{}_j \partial_\mu(\epsilon(x)\Psi^j(x)).$$

$\mathcal{L}_1$ transforms as:

$$\delta\mathcal{L}_1 = \frac{\partial \mathcal{L}_1}{\partial \Phi^n} i\epsilon t^n{}_m \Phi^m + \frac{\partial \mathcal{L}_1}{\partial \Psi^i} i\epsilon \tau^i{}_j \Psi^j.$$

Invariance of the Lagrangian density implies (here ϵ is no longer a function of spacetime):

$$\begin{aligned} 0 =& \frac{\partial \mathcal{L}_0}{\partial \Phi^n} i\epsilon t^n{}_m \Phi^m + \frac{\partial \mathcal{L}_0}{\partial \Psi^i} i\epsilon \tau^i{}_j \Psi^j + \frac{\partial \mathcal{L}_0}{\partial \partial_\mu \Phi^n} i t^n{}_m \epsilon \partial_\mu \Phi^m(x) + \frac{\partial \mathcal{L}_0}{\partial \partial_\mu \Psi^i} i\tau^i{}_j \epsilon \partial_\mu \Psi^j(x) \\ &+ \frac{\partial \mathcal{L}_1}{\partial \Phi^n} i\epsilon t^n{}_m \Phi^m + \frac{\partial \mathcal{L}_1}{\partial \Psi^i} i\epsilon \tau^i{}_j \Psi^j. \end{aligned}$$

The variation of the action is thus given by:

$$\delta I[\Phi, \Psi] = \int \mathrm{d}^4 x \left[\frac{\partial \mathcal{L}_0}{\partial \partial_\mu \Phi^n} i t^n{}_m \Phi^m(x) + \frac{\partial \mathcal{L}_0}{\partial \partial_\mu \Psi^i} i\tau^i{}_j \Psi^j(x) \right] \partial_\mu \epsilon(x).$$

Comparison with (7.3.4) shows:

$$J^\mu = -i \left[\frac{\partial \mathcal{L}_0}{\partial \partial_\mu \Phi^n} t^n{}_m \Phi^m(x) + \frac{\partial \mathcal{L}_0}{\partial \partial_\mu \Psi^i} \tau^i{}_j \Psi^j(x) \right].$$

Chapter 8

Problem 2

Carry out the canonical quantization of the theory of a charged scalar field Φ and its interaction with electromagnetism, with Lagrangian density:

$$\mathcal{L} = -(D_\mu \Phi)^\dagger (D^\mu \Phi) - m^2 \Phi^\dagger \Phi - \lambda(\Phi^\dagger \Phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

where

$$D_\mu \Phi \equiv \partial_\mu \Phi - iqA_\mu \Phi, \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Use Coulomb gauge. Express the Hamiltonian in terms of the fields A, Φ, and $\Phi^\dagger$ and their canonical conjugates. Evaluate the interaction $V(t)$ in the interaction-picture in terms of the interaction picture fields and their derivatives.

Solution

The Lagrangian is given by:

$$\begin{aligned}
&\mathcal{L} = -(D_\mu \Phi)^\dagger (D^\mu \Phi) - m^2 \Phi^\dagger \Phi - \lambda(\Phi^\dagger \Phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\
&D_\mu \Phi = \partial_\mu \Phi - iqA_\mu \Phi, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu
\end{aligned}$$

Coulomb gauge gives:

$$\begin{aligned}
&\boldsymbol{\nabla} \cdot \mathbf{A} = 0 \\
&A^0(\mathbf{x}, t) = \int \mathrm{d}^3 y \, \frac{J^0(\mathbf{y}, t)}{4\pi |\mathbf{x} - \mathbf{y}|}
\end{aligned}$$

The conserved current is given by:

$$\begin{aligned}
J^\mu(x) &= \sum_l \frac{\partial \mathcal{L}_M}{\partial D_\mu \Psi^l} \Big[-iq_l \Psi^l \Big] \\
&= -i \Big[\frac{\partial \mathcal{L}_M}{\partial D_\mu \Phi} q\Phi + \frac{\partial \mathcal{L}_M}{\partial D_\mu \Phi^\dagger} \Phi^\dagger(-q) \Big] \\
&= -i \Big[(-\partial_\mu \Phi^\dagger - iqA^\mu \Phi^\dagger) q\Phi + (-\partial_\mu \Phi + iqA^\mu \Phi) \Phi^\dagger (-q) \Big] \\
&= iq \Big[\Phi \partial^\mu \Phi^\dagger - \Phi^\dagger \partial^\mu \Phi \Big] - 2q^2 A^\mu \Phi^\dagger \Phi
\end{aligned}$$

The Hamiltonian is given by:

$$\begin{aligned}
&H = \int \mathrm{d}^3 x \left[\frac{1}{2} \mathbf{\Pi}_\perp^2 + \frac{1}{2} (\boldsymbol{\nabla} \times \mathbf{A})^2 - \mathbf{J} \cdot \mathbf{A} + \frac{1}{2} J^0 A^0 \right] \\
&\mathbf{J} = iq \Big[\Phi \boldsymbol{\nabla} \Phi^\dagger - \Phi^\dagger \boldsymbol{\nabla} \Phi \Big] - 2q^2 \mathbf{A} \Phi^\dagger \Phi, \quad J^0 = iq \Big[\Phi \partial^0 \Phi^\dagger - \Phi^\dagger \partial^0 \Phi \Big] - 2q^2 A^0 \Phi^\dagger \Phi.
\end{aligned}$$

$V(t)$ in the interaction picture in terms of interaction fields and derivatives:

$$V(t) = \exp(iH_0 t)V(t=0)\exp(-iH_0 t)$$

By (8.4.23):

$$\begin{aligned} V(t) &= -\int \mathrm{d}^3x\, j_\mu(\mathbf{x},t)a^\mu(\mathbf{x},t) + V_{\mathrm{Coul}}(t) + V_{\mathrm{Matter}}(t) \\ &= -\int \mathrm{d}^3x\, \mathbf{j}\cdot\mathbf{a} + \int \mathrm{d}^3x\,\mathrm{d}^3y\, \frac{j^0(\mathbf{x},t)j^0(\mathbf{y},t)}{4\pi|\mathbf{x}-\mathbf{y}|} - \int \mathrm{d}^3x\, \lambda(\Phi^\dagger(t)\Phi(t))^2 \end{aligned}$$

where

$$\begin{aligned} &\Phi(t) = \exp(iH_0 t)\Phi(t=0)\exp(-iH_0 t) \\ &\mathbf{a}(t) = \exp(iH_0 t)\mathbf{A}(t=0)\exp(-iH_0 t) \\ &\mathbf{j} = iq\Big[\Phi(t)\boldsymbol{\nabla}\Phi^\dagger(t) - \Phi^\dagger(t)\boldsymbol{\nabla}\Phi(t)\Big] - 2q^2\mathbf{a}(t)\Phi^\dagger(t)\Phi(t) \\ &j^0 = iq\Big[\Phi(t)\partial^0\Phi^\dagger(t) - \Phi^\dagger(t)\partial^0\Phi(t)\Big]. \end{aligned}$$

The commutation relations are given by:

$$\begin{aligned} &\big[a^i(\mathbf{x},t), \pi^i(\mathbf{y},t)\big] = i\Big[\delta_{ij}\delta^3(\mathbf{x}-\mathbf{y}) + \frac{\partial^2}{\partial x^i \partial x^j}\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\Big] \\ &\big[a^i(\mathbf{x},t), a^j(\mathbf{y},t)\big] = 0 \\ &\big[\pi^i(\mathbf{x},t), \pi^i(\mathbf{y},t)\big] = 0 \\ &\Big[\Phi(\mathbf{x},t), \dot{\Phi}^\dagger(\mathbf{y},t)\Big] = i\delta^3(\mathbf{x}-\mathbf{y}) \\ &[\Phi(\mathbf{x},t), \Phi(\mathbf{y},t)] = 0 \end{aligned}$$

Chapter 9

Problem 1

> Consider a non-relativistic particle of mass m, moving along the x-axis in a potential $V(x) = \frac{1}{2}m\omega^2 x^2$. Use path-integral methods to find the probability that if the particle is at x_1 at time t_1, then it is between x and $x + \mathrm{d}x$ at time t.

Solution

The dynamics of a non-relativistic quantum-mechanical system are governed by the Hamiltonian

$$H(\hat{x}, \hat{p}) = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

where $\hat{x}$ and $\hat{p}$ satisfy the canonical commutation relation $[\hat{x}, \hat{p}] = i$. Let $|x\rangle$ denote eigenstates of $\hat{x}$. Given that the system is initially in the state $|\psi(t_1)\rangle = |x_1\rangle$ we want to compute the probability $P(x,t)\mathrm{d}x$ that the oscillator will be measured at between x and $x + \mathrm{d}x$ at a later time t. We have

$$P(x,t) = |\langle x|\psi(t)\rangle|^2 = \left|\langle x| e^{-iH(t-t_1)} |\psi(t_1)\rangle\right|^2 = \left|\langle x| e^{-iH(t-t_1)} |x_1\rangle\right|^2.$$

The quantity of interest is therefore $\langle x| e^{-iH(t-t_1)} |x_1\rangle$.

<u>Free particle:</u> Before tackling the quantum harmonic oscillator problem, first note that for a free particle ($V(\hat{x}) = 0$) the propagator is given by a straightforward direct computation as

$$\langle x| e^{-iH_0(t-t_1)} |x_1\rangle = \left(\frac{m}{2\pi i(t-t_1)}\right)^{1/2} \exp\left(\frac{im(x-x_1)^2}{2(t-t_1)}\right) \tag{9-1-1}$$

<u>Quantum harmonic oscillator:</u> For the harmonic oscillator we have $V(\hat{x}) = \frac{1}{2}m\omega^2\hat{x}^2$. In order to use the path-integral methods developed in 9.1 we define eigenstates $|x;t\rangle$ and $|p;t\rangle$ of the *Heisenberg-picture* operators $\hat{x}(t) \equiv e^{iH(t-t_1)}\hat{x}e^{-iH(t-t_1)}$ and $\hat{p}(t) \equiv e^{iH(t-t_1)}\hat{p}e^{-iH(t-t_1)}$ as

$$|x;t\rangle \equiv e^{iH(t-t_1)} |x\rangle, \quad |p;t\rangle \equiv e^{iH(t-t_1)} |p\rangle$$

We then find that

$$\langle x| e^{-iH(t-t_1)} |x_1\rangle = \underbrace{\langle x;t| e^{iH(t-t_1)}}_{\langle x|} e^{-iH(t-t_1)} \underbrace{|x_1;t_1\rangle}_{|x_1\rangle} = \langle x;t|x_1;t_1\rangle. \tag{9-1-2}$$

The expression in the final equality is precisely of the form of the lhs of (9.1.34) so we can immediately use the results of chapter 9 to compute it. First, since H is quadratic in p, the discussion in section 9.3 allows us to replace the argument of the exponential in (9.1.34) with i times the action, $S[x(\tau)] = \int_{t_1}^{t} \mathrm{d}\tau (m\dot{x}^2/2 - m\omega^2 x^2/2)$ *after* evaluating the integrals over the momenta:

$$\begin{aligned}\langle x;t|x_1;t_1\rangle &= \int \left[\prod_\tau \mathrm{d}x(\tau)\right]\left[\prod_\tau \frac{\mathrm{d}p(\tau)}{2\pi}\right] \times \\ &\qquad \exp\left\{ i\int_{t_1}^{t} \mathrm{d}\tau \left[p(\tau)\dot{x}(\tau) - \frac{p^2(\tau)}{2m} - \frac{1}{2}m\omega^2 x^2(\tau)\right]\right\} \\ &\propto \int \left[\prod_\tau \mathrm{d}x(\tau)\right] \exp\left\{ i\int_{t_1}^{t} \mathrm{d}\tau \left[\frac{1}{2}m\dot{x}^2(\tau) - \frac{1}{2}m\omega^2 x^2(\tau)\right]\right\}.\end{aligned}$$

Using the techniques given in Chapter 3-11 of [1] we find that this is equal to

$$C\left(\frac{\omega T}{\sin(\omega T)}\right)^{1/2} e^{iS_{\text{classical}}}$$

where C is a constant independent of ω, $T \equiv t - t_1$ and $S_{\text{classical}}$ is the action $S[x(\tau)]$ evaluated at $x(\tau) = \bar{x}(\tau)$, the solution to the classical equations of motion (i.e. $\bar{x}(\tau) = A\cos(\omega\tau) + B\sin(\omega\tau)$ with A, B chosen so that $\bar{x}(t) = x$ and $\bar{x}(t_1) = x_1$.) Explicitly,

$$S_{\text{classical}} = \frac{m\omega}{2\sin\omega T}\left[(x^2 + x_1^2)\cos\omega T - 2x\,x_1\right]. \tag{9-1-3}$$

In the limit $\omega \to 0$ our result must reduce to that of a free particle. We have

$$C\left(\frac{\omega T}{\sin(\omega T)}\right)^{1/2} e^{iS_{\text{classical}}} \to C\exp\left(\frac{im(x-x_1)^2}{2T}\right). \tag{9-1-4}$$

Comparison with eq. (9-1-1) tells us that C must equal $[m/2\pi iT]^{1/2}$. So, finally*

$$P(x,t)\,\mathrm{d}x = |\langle x;t|x_1;t_1\rangle|^2\,\mathrm{d}x = \frac{m\omega}{2\pi\sin\omega T}\,\mathrm{d}x.$$

*Thank you to Jiangyuan Qian for correcting a mistake I (Ray) made in evaluating $|\langle x;t|x_1;t_1\rangle|^2$ in an earlier version of these notes.

Problem 4

The Lagrangian density of the free spin 3/2 Rarita-Schwinger field is

$$\mathcal{L} = -\bar{\psi}^\mu(\gamma^\nu\partial_\nu + m)\psi_\mu - \frac{1}{3}\bar{\psi}^\mu(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu)\psi^\nu + \frac{1}{3}\bar{\psi}^\mu\gamma_\mu(\gamma^\sigma\partial_\sigma - m)\gamma^\nu\psi_\nu.$$

Use path-integral methods to find the propagator of this field.

Solution

References

[1] R.P. Feynman, A.R. Hibbs, and D.F. Styer. *Quantum Mechanics and Path Integrals*. Dover Books on Physics. Dover Publications, 2010.