1. Centroid of a triangle

I recently spent a few days trying to understand some well-known results about triangles and their centroids. First, there is the famous centroid theorem. Consider a triangle with vertices A, B, C. Construct the medians by drawing lines from the midpoints A', B', C' to the opposite vertex (A, B, C respectively).

Then, several properties can be shown:

1. *The three medians intersect at a point G.*
2. *The point G is the centroid.*
3. *Its coordinates are given by $\vec{r}_G = \frac{1}{3}(\vec{r}_A + \vec{r}_B + \vec{r}_C)$.*
4. *G splits each median into segments with a 2:1 ratio.*

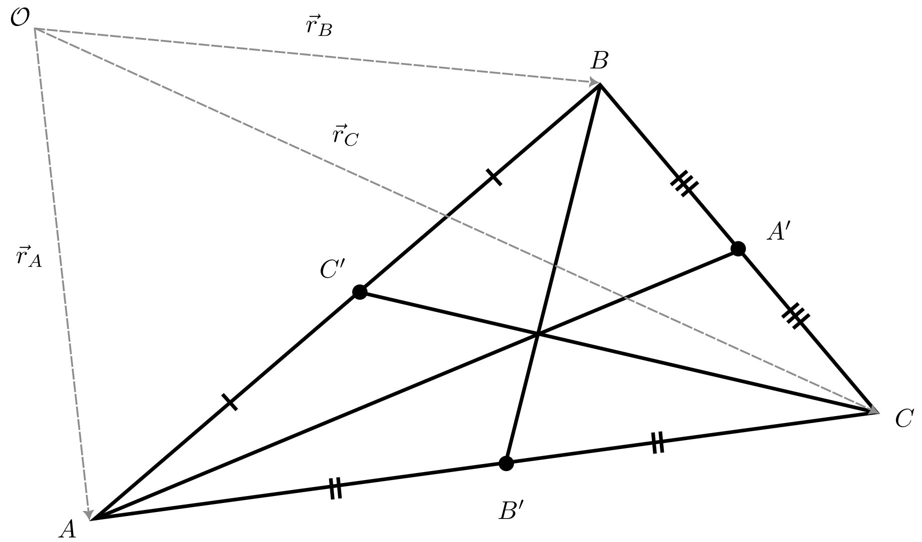

Figure 1: Diagram of triangle showing vertices A, B, C, and midpoints A', B', C'.

Let's go ahead and prove these statements one by one.

1.1. The three medians intersect at a common point

Let's show this explicitly by finding where two medians intersect and show that the intersection of another distinct pair is the same.

First, define the linearly independent vectors

$$\begin{aligned} \vec{u} &\equiv \vec{r}_{AC} \equiv \vec{r}_C - \vec{r}_A, \quad \text{and} \\ \vec{v} &\equiv \vec{r}_{AB} \,. \end{aligned} \tag{1.1.1}$$

The intersection G is along the line $\overline{BB'}$ so we must have

$$\vec{r}_{AG} \equiv \lambda_1 \vec{r}_{AA'} = \lambda_1 \left(\frac{\vec{u} + \vec{v}}{2} \right) \quad \text{for some } 0 < \lambda_1 < 1 \,. \tag{1.1.2}$$

Similarly, for $\overline{AA'}$ we have

$$\vec{r}_{BG} = \lambda_2 \vec{r}_{BB'} = \lambda_2 \left(\frac{\vec{u}}{2} - \vec{v} \right) \text{ for some } 0 < \lambda_2 < 1 \,. \tag{1.1.3}$$

We can also write $\vec{r}_{BG} = \vec{r}_{AG} - \vec{r}_{AB} = \vec{r}_{GA} - \vec{v}$ so that Equation (1.1.3) can be rewritten as

$$\vec{r}_{AG} = \frac{\lambda_2}{2} \vec{u} + (1 - \lambda_2) \vec{v} \,. \tag{1.1.4}$$

Combining Equation (1.1.2) and Equation (1.1.4) yields

$$\frac{\lambda_1}{2}\vec{u} + \frac{\lambda_1}{2}\vec{v} = \frac{\lambda_2}{2}\vec{u} + (1-\lambda_2)\vec{v}\,. \tag{1.1.5}$$

Since $\vec{u}$ and $\vec{v}$ are linearly independent their coefficients must separately be equal. This gives us two equations with two unknowns:

$$\begin{cases} \frac{\lambda_1}{2} = \frac{\lambda_2}{2} \\ \frac{\lambda_1}{2} = (1-\lambda_2) \quad , \end{cases} \tag{1.1.6}$$

which can be solved to give ($\lambda_1 = \lambda_2 = \frac{2}{3}$). Hence, we obtain

$$\vec{r}_G = \frac{1}{3}(\vec{r}_A + \vec{r}_B + \vec{r}_C)\,. \tag{1.1.7}$$

Note how this answer is symmetric with respect to A, B, and C. We can repeat the argument for any other pair of medians and reach the same conclusion. Therefore all three medians intersect at a point, and the intersections coordinates is the arithmetic mean of the coordinates of A, B, C .

1.2. The point G is the centroid

We can establish this fact by the definition of the centroid as the average position of all points in the polygon. Namely,

$$\vec{r}_{\text{cm}} = \frac{1}{\mathcal{A}}\iint_{\mathcal{A}} \vec{r}\ \mathrm{d}\mathcal{A}\,, \tag{1.2.1}$$

where $\mathcal{A}$ is the area of the shape and "cm" is short for center of mass. (This assumes uniform density.) The purpose of this section is to demonstrate that $\vec{r}_{\text{cm}} = \vec{r}_G$ where $\vec{r}_G$ is given by Equation (1.1.7).

The most economic way to demonstrate this is to use the coordinate system given by vectors $\vec{u}$ and $\vec{v}$ which were defined in Equation (1.1.1), and use Green's theorem to evaluate the double integral.

To proceed, note that $2\mathcal{A}\,\mathrm{d}u\,\mathrm{d}v = \mathrm{d}x\,\mathrm{d}y$ (this is because) $\mathrm{d}u\,\mathrm{d}v$ form a parallelogram as seen in the xy-plane and $(u,v) \in [0,1] \times [0,1]$ is a parallelogram with twice the area of our triangle.

In the uv coordinate system $r = u\vec{u} + v\vec{v} \equiv \binom{u}{v}$. Hence,

$$\vec{r}_{\text{cm}} = \frac{1}{\mathcal{A}}\iint_{\mathcal{A}} \begin{pmatrix} u \\ v \end{pmatrix} (2\mathcal{A})\,\mathrm{d}u\,\mathrm{d}v\,. \tag{1.2.2}$$

Green's theorem states that the double integral can be expressed as a line integral over the boundary. In particular,

$$\oint_{\partial\mathcal{A}} P\,\mathrm{d}u + Q\,\mathrm{d}v = \iint_{\mathcal{A}} \frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\,\mathrm{d}u\,\mathrm{d}v\,. \tag{1.2.3}$$

In the first row of Equation (1.2.2) take $Q = u^2/2$. In the second row take $P = -v^2/2$. Then,

$$\vec{r}_{\text{cm}} = 2\oint_{\partial\mathcal{A}} \begin{pmatrix} \frac{1}{2}u^2\,\mathrm{d}v \\ -\frac{1}{2}v^2\,\mathrm{d}u \end{pmatrix} = 2(I_1 + I_2 + I_3). \tag{1.2.4}$$

We can break up the path $\partial\mathcal{A}$ into 3 segments which we'll call $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ (see Figure 2).

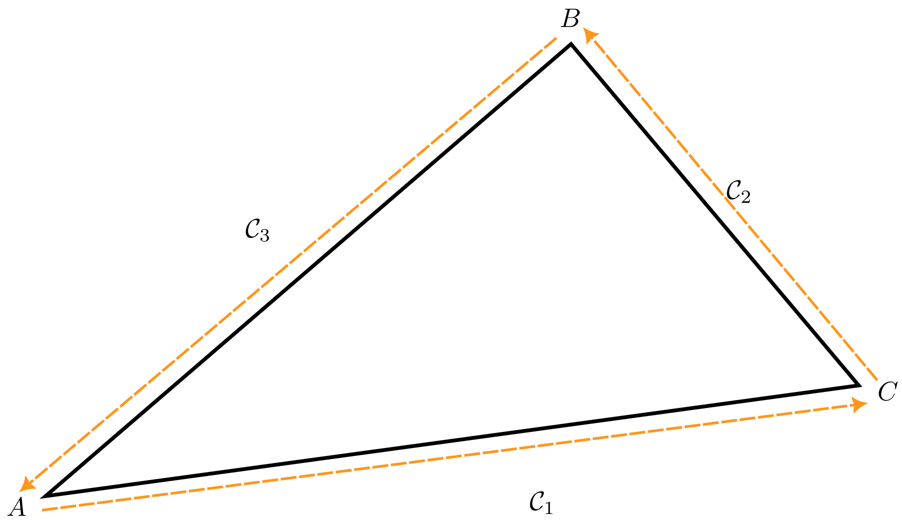

Figure 2: Integration contour. Integral I_i is taken along curve $\mathcal{C}_i$ as indicated.

We'll choose $\mathcal{C}_1$ to be along $\overline{AC}$, $\mathcal{C}_2$ to be along $\overline{CB}$, and $\mathcal{C}_3$ to be along $\overline{CA}$. We can label the corresponding integrals I_1, I_2, and, I_3 respectively. What can we say about these integrals right off the bat? We know that along $\mathcal{C}_1$ $v = 0$ and u goes from 0 to 1. Both the top and bottom rows vanish. So, $I_1 = 0$. For I_3 we have $v = 0$ and u goes from 1 to 0. Again, I_2 is trivially 0. The only nonzero contribution comes from I_3, which we are ready to evaluate explicitly.

$$\begin{aligned} I_3 &= \int_{(1,0)}^{(0,1)} \begin{pmatrix} \frac{1}{2}u^2 \, \mathrm{d}v \\ -\frac{1}{2}v^2 \, \mathrm{d}u \end{pmatrix} \\ &= \int_{(0,0)}^{(1,1)} \begin{pmatrix} \frac{1}{2}u^2 \, \mathrm{d}v \\ \frac{1}{2}v^2 \, \mathrm{d}u \end{pmatrix} \end{aligned} \tag{1.2.5}$$

The first row is

$$\int_0^1 \frac{1}{2}(1-v)^2 \, \mathrm{d}v = \frac{1}{6} \,, \tag{1.2.6}$$

and similar for the second row. Thus,

$$\vec{r}_{\mathrm{cm}} = \frac{1}{3}\vec{u} + \frac{1}{3}\vec{v} \tag{1.2.7}$$

as seen in the uv coordinate system (whose origin is at A). In terms of the original xy cartesian coordinates we need to translate $\vec{r}_{\mathrm{cm}} \to \vec{r}_{\mathrm{cm}} - \vec{r}_A$ so we get the matching expression

$$\vec{r}_{\mathrm{cm}} = \frac{1}{3}(\vec{r}_A + \vec{r}_B + \vec{r}_C) = \vec{r}_G \,. \tag{1.2.8}$$

Which is the result we wanted!

1.3. G splits each median into a 2:1 ratio

This one is actually pretty easy to show now that we have already proved that the medians intersect at a common point. First, notice that if you divide a triangle into two triangles by drawing a median, the two triangles must have the same area because they have the same base (given by each segment formed by the median), and the same height (since the tallest point is at the vertex farthest from the base.) With this in mind let's label each of the areas for each triangle formed by drawing all the medians as in Figure 3.

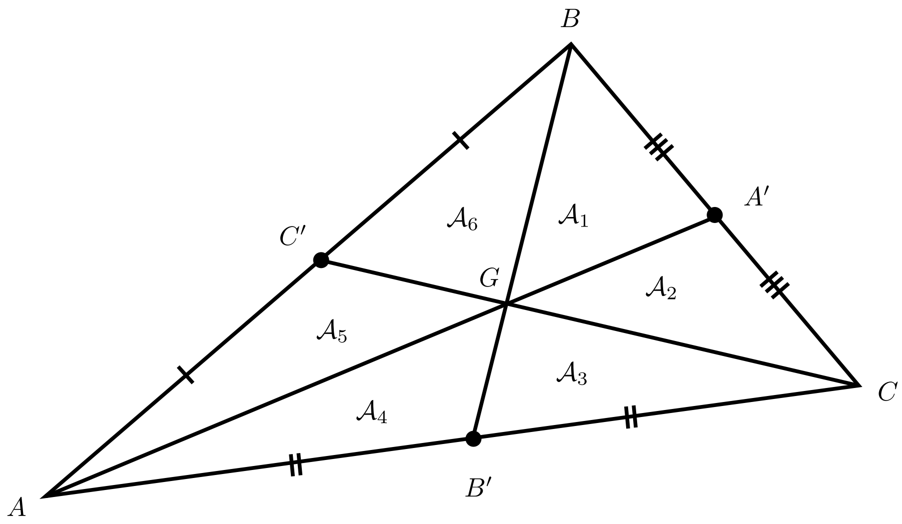

Figure 3: Triangle areas

We immediately get the following relations:

$$\begin{cases} \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 = \mathcal{A}_4 + \mathcal{A}_5 + \mathcal{A}_6 & (\text{from } \overline{BB'}) \\ \mathcal{A}_6 + \mathcal{A}_1 + \mathcal{A}_2 = \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{A}_5 & (\text{from } \overline{CC'}) \\ \mathcal{A}_5 + \mathcal{A}_6 + \mathcal{A}_1 = \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 & (\text{from } \overline{AA'})\,. \end{cases} \tag{1.3.1}$$

This can be solved to give $\mathcal{A}_1 = \mathcal{A}_4, \mathcal{A}_2 = \mathcal{A}_5, \mathcal{A}_3 = \mathcal{A}_6$. Furthermore, notice that $\triangleright A'BG$ and $\triangleright A'CG$ have the same base and height, so their areas are identical. Similar arguments for the other triangles prove that they all have the same area. That is, $\mathcal{A}_i = \mathcal{A}/6$ for all i.

We can use this fact to prove that G splits the medians into 2:1 segments. Consider $\triangleright ABG$ and $\triangleright ABB'$. Their areas are $\mathcal{A}_{ABG} = A/3$ and $\mathcal{A}_{ABB'} = A/2$. Moreover, in terms of their base and height we have,

$$\begin{aligned} \mathcal{A}_{ABG} &= \frac{1}{2}\,\overline{BG}\,h = \frac{A}{3} \\ \mathcal{A}_{ABB'} &= \frac{1}{2}\,\overline{BB'}\,h = \frac{A}{2}\,. \end{aligned} \tag{1.3.2}$$

Taking the ratio we get

$$\frac{\mathcal{A}_{ABG}}{\mathcal{A}_{ABB'}} = \frac{\overline{BG}}{\overline{BB'}} = \frac{2}{3} \tag{1.3.3}$$

In other words, $\overline{BG}$ is two-thirds of the full median and $\overline{GB'}$ is one-third. So the segments are in a 2:1 ratio. The argument is identical for the other medians.